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Solution of the Heisenberg equations for an atom interacting with radiation

B Buck and C V Sukumar

Department of Theoretical Physics, 1 Keble Road, Oxford, UK

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Abstract. It is shown that the nonlinear equation of motion for the energy operator of a few-level atom interacting with a single mode radiation field can be solved explicitly.

1. Introduction

The model system of an idealised few-level atom, or a spin system in a magnetic field, interacting with a single mode radiation field is of particular interest in quantum optics. Here we consider an atom with $2J+1$ levels and represent its Hamiltonian and transition operators by means of the components of an angular momentum vector operator \mathbf{J} . The field is described by the creation and destruction operators for a linear oscillator and the atom-field coupling is assumed to induce transitions between neighbouring atomic levels with emission or absorption of single radiation quanta.

This type of model is considered to contain at least some of the essential physics underlying the interaction of matter with radiation and has been widely studied (Jaynes and Cummings 1963, Senitzky 1971), especially for two-level systems (Allen and Eberly 1975). The two-state atom corresponds to $J = \frac{1}{2}$ and for given initial conditions the dynamical behaviour of the atomic energy can be found completely in either the Schrödinger or Heisenberg pictures. For systems with more than two levels ($J > \frac{1}{2}$) it has previously been thought that the Heisenberg equations were not amenable to formal solution; but we will show that these nonlinear operator equations can in fact be solved explicitly, even when the radiation is not in resonance with the atom.

In § 2 we give the Hamiltonian and discuss the relationship between the Schrödinger and Heisenberg methods. In § 3 we derive the equation of motion for the energy operator of the atom and solve it for a two-level system on or off resonance. We then describe a method of reducing the equations for $J > \frac{1}{2}$ to linear form and apply it to give the complete solution for a three-level atom in resonant interaction with the field. In § 4 we derive the solution for a three-level atom detuned from resonance and also extend the methods to treat a four-state resonant system. Section 5 contains a short summary.

2. Model Hamiltonian

The energy operator for an atomic system with $2J+1$ equally spaced levels, with separation ε , is represented by εJ_3 where J_3 is the three-axis component of angular

momentum. A single quantum of the radiation mode is taken to have unit energy so that the field Hamiltonian is $a^\dagger a$, the number operator for oscillator phonons. For the atom-radiation coupling we assume a simplified version of electrodynamic dipole coupling, $(\frac{1}{2}\lambda)[J_+a + J_-a^\dagger]$, where λ is a constant and $J_\pm = (J_1 \pm iJ_2)$ are the angular momentum shift operators. This describes emission or absorption of a single phonon accompanied by the lowering or raising of the atomic energy to an adjacent level. It is known as the rotating wave approximation and appears to be essential for obtaining a soluble model. The simplified interaction implies neglect of small energy shifts and is physically reasonable if the field is not too intense.

With these assumptions the Hamiltonian is

$$H = a^\dagger a + \varepsilon J_3 + (\frac{1}{2}\lambda)[J_+a + J_-a^\dagger], \quad (1)$$

and at a fixed time the relevant commutation relations are

$$[J_+, J_-] = 2J_3, [J_3, J_\pm] = \pm J_\pm, [a, a^\dagger] = 1, \quad (2)$$

with all other equal time commutators zero.

It is easy to establish that the operators $J^2 = \frac{1}{2}[J_+J_- + J_-J_+] + J_3^2$ and $C = a^\dagger a + J_3$ commute with H and each other; hence simultaneous eigenstates of H , J^2 and C may be constructed. For a fixed J -value the atomic states are denoted by $|M\rangle$, $M = -J, -J+1, \dots, +J$ and satisfy $J_3|M\rangle = M|M\rangle$ while the field states $|n\rangle$, $n = 0, 1, 2, \dots$ are eigenstates of the phonon number operator with $a^\dagger a|n\rangle = n|n\rangle$. The eigenstates of H may be expanded in the form

$$|E, \gamma\rangle = \sum_{M,n} \delta_{\gamma, M+n} a(M, n) |M\rangle |n\rangle, \quad (3)$$

and are labelled by the energy E and the eigenvalues $\gamma = M + n$ of the operator C .

In the Schrödinger picture the expectation value $\mathcal{E}(t)$ of the atomic energy in a state $|\psi(t)\rangle$, which has evolved from some given initial state $|\psi(0)\rangle$ of atom and field, is given by

$$\mathcal{E}(t) = \varepsilon \langle \psi(t) | J_3 | \psi(t) \rangle. \quad (4)$$

To evaluate this we need first to find the eigenvalues and eigenstates of H , then to construct the initial state as a superposition

$$|\psi(0)\rangle = \sum_{E,\gamma} \alpha(E, \gamma) |E, \gamma\rangle, \quad (5)$$

and finally to calculate the complicated sum

$$\mathcal{E}(t) = \varepsilon \sum_{E,E',\gamma} \alpha^*(E', \gamma) \alpha(E, \gamma) \exp[i(E' - E)t] \langle E', \gamma | J_3 | E, \gamma \rangle. \quad (6)$$

In the Heisenberg picture the initial state remains fixed and the operators are time-dependent. Hence if we can determine, once and for all, the explicit time dependence of $J_3(t) = \exp(iHt)J(0)\exp(-iHt)$ we can follow the evolution of $\mathcal{E}(t)$ by calculating the matrix element

$$\mathcal{E}(t) = \varepsilon \langle \psi(0) | J_3(t) | \psi(0) \rangle \quad (7)$$

for any desired initial state. It is clear from equation (6) that the explicit expression for $J_3(t)$ must contain effective frequency operators whose eigenvalues are all the energy differences between the eigenstates of the coupled system. Thus we expect the

solution for $J_3(t)$ to be quite complicated. Indeed the Heisenberg method for this model leads to nonlinear operator equations and these are usually considered to be intractable.

3. Two- and three-level atoms

3.1. Heisenberg equations

Starting from the Hamiltonian (1) and the commutators (2) we write down the Heisenberg equations for various operators Q in the usual way, i.e. $i\dot{Q}=[Q, H]$. It is convenient to define the constant

$$\alpha = \varepsilon - 1, \tag{8}$$

measuring the detuning from resonance, and the subsidiary operators

$$A = H - C, \quad B = (\frac{1}{2}\lambda)[J_+a - J_-a^\dagger] \tag{9a, b}$$

$$C = a^\dagger a + J_3, \quad D = (\frac{1}{2}\lambda)[J_+a + J_-a^\dagger] = A - \alpha J_3. \tag{9c, d}$$

The operator C commutes with H and is therefore a constant of the motion which, moreover, commutes separately with $a^\dagger a$ and J_3 . Clearly A is also a constant operator.

Using the relations

$$J_+J_- = \mathbf{J}^2 - J_3^2 + J_3, \quad J_-J_+ = \mathbf{J}^2 - J_3^2 - J_3, \tag{10}$$

the following Heisenberg equations are quickly established:

$$i\dot{J}_3 = B, \tag{11}$$

$$i\dot{B} = \Omega^2 J_3 - (\frac{1}{2}\lambda^2)[3J_3^2 - \mathbf{J}^2] - \alpha A, \tag{12}$$

in which the frequency operator Ω is defined by

$$\Omega^2 = \alpha^2 + \lambda^2(C + \frac{1}{2}). \tag{13}$$

These expressions lead immediately to the nonlinear operator equation of motion

$$\ddot{J}_3 + \Omega^2 J_3 = (\frac{1}{2}\lambda^2)[3J_3^2 - \mathbf{J}^2] + \alpha A, \tag{14}$$

which involves only the atomic energy operator J_3 and the constant operators Ω^2 , \mathbf{J}^2 and A . Note that Ω^2 and \mathbf{J}^2 commute with J_3 and B at all times while $A = H - C$ does not.

3.2. Two-level atom

The above equations are easily solved for a two-level atom both on and off resonance. For a system with $J = \frac{1}{2}$ we always have $\mathbf{J}^2 = \frac{3}{4}$ and $J_3^2 = \frac{1}{4}$ so that the operator equation (14) reduces to

$$\ddot{J}_3 + \Omega^2 J_3 = \alpha A, \tag{15}$$

and the general solution of this is easily seen to be

$$J_3(t) = [J_3(0) - (\alpha/\Omega^2)A] \cos \Omega t - i[B(0)/\Omega] \sin \Omega t + (\alpha/\Omega^2)A. \tag{16}$$

This solution has been used by Eberly *et al* (1980, 1981) to study the long time behaviour of $\langle \psi(0) | J_3(t) | \psi(0) \rangle$ for an atom, initially in its ground state, interacting

with a coherent radiation field, and it led to the surprising discovery of periodic decay and regeneration phenomena. Buck and Sukumar (1981) used a related model, with a solution similar to the above, to find the behaviour of $\langle J_3(t) \rangle$, at arbitrary times, in closed analytic form, for both coherent and thermal radiation.

3.3. Extension of method

The above simplification does not occur for an atom with more than two states ($J > \frac{1}{2}$). To make further progress we return to equation (14) and differentiate it twice with respect to time to obtain

$$\begin{aligned} \ddot{J}_3 + \Omega^2 \check{J}_3 &= \frac{3}{2} \lambda^2 (d^2/dt^2)(J_3^2) \\ &= \frac{3}{2} \lambda^2 (d/dt)[J_3 \dot{J}_3 + \dot{J}_3 J_3] \\ &= (\frac{3}{2} \lambda^2) [\{J_3, \check{J}_3\} - 2B^2], \end{aligned} \tag{17}$$

where $\{ , \}$ denotes an anticommutator and we have used equation (11). Now from the definitions of B and D we have

$$B^2 = D^2 - (\frac{1}{2} \lambda^2) [J_+ J_- a a^\dagger + J_- J_+ a^\dagger a], \tag{18}$$

and using equations (10), together with $D = A - \alpha J_3$, we find

$$B^2 = \Omega^2 J_3^2 - \alpha \{A, J_3\} + \lambda^2 J_3 (\mathbf{J}^2 - J_3^2 - \frac{1}{2}) - \Gamma, \tag{19}$$

where Γ is a constant operator defined by

$$\Gamma = \lambda^2 (C + \frac{1}{2}) \mathbf{J}^2 - A^2. \tag{20}$$

The anticommutator $\{J_3, \check{J}_3\}$ may be rewritten in terms of J_3 using equation (14) so that

$$\ddot{J}_3 + \Omega^2 \check{J}_3 = (\frac{3}{2} \lambda^2) [3\alpha \{A, J_3\} - 4\Omega^2 J_3^2 + \lambda^2 J_3 (5J_3^2 - 3\mathbf{J}^2 + 1) + 2\Gamma]. \tag{21}$$

Finally, we substitute for J_3^2 from equation (14) and simplify to finish with

$$\begin{aligned} \ddot{J}_3 + 5\Omega^2 \check{J}_3 + 4\Omega^4 J_3 &= (\frac{3}{2} \lambda^2) [\lambda^2 J_3 (5J_3^2 - 3\mathbf{J}^2 + 1) + 3\alpha \{A, J_3\}] \\ &\quad + [4\alpha \Omega^2 A + \lambda^2 (\Omega^2 - 3\alpha^2) \mathbf{J}^2 - 3\lambda^2 A^2], \end{aligned} \tag{22}$$

where the last bracketed term is a constant of the motion.

It may seem at first sight that not much has been achieved by these manipulations since we started from a second-order equation for J_3 with a quadratic nonlinearity and have arrived at a fourth-order equation with a term cubic in J_3 . However, the equation is now in a form suitable for treating the three-level atom since we can use the operator identity $J_3^3 = J_3$, valid for systems with $J = 1$. Hence for this case equation (22) is actually linear in J_3 and can be solved.

This suggests a systematic technique for solution of the equations when $J \geq 1$. Continuing the processes of differentiation and simplification we generate higher-order equations with increasing degrees of nonlinearity. The system of equations eventually leads to a linear form because the general identity, valid for any fixed J ,

$$(J_3 - J)(J_3 - (J - 1)) \dots (J_3 + (J - 1))(J_3 + J) = 0 \tag{23}$$

allows the operator J_3^{2J+1} to be expressed in terms of lower powers of J_3 . Unfortunately, the amount of work involved escalates rapidly as J increases and the general linear

equation for arbitrary J has not yet been found. Here we content ourselves with exhibiting the solutions for three- and four-state atoms.

3.4. Resonant three-level atom

For a three-level atom in resonance with the field we have $J = 1$ and $\alpha = \varepsilon - 1 = 0$. The equation of motion (22) then simplifies to

$$\ddot{J}_3 + 5\Omega^2 \dot{J}_3 + 4\Omega^4 J_3 = \lambda^2 [2\Omega^2 - 3A^2], \tag{24}$$

where now $\Omega^2 = \lambda^2 (C + \frac{1}{2})$ and we have used the relations $J^2 = 2$, $J_3^3 = J_3$. Thus the operator

$$\tilde{J}_3 = J_3 - (\lambda^2/4\Omega^4)[2\Omega^2 - 3A^2] \tag{25}$$

satisfies the homogeneous linear equation

$$\ddot{\tilde{J}}_3 + 5\Omega^2 \dot{\tilde{J}}_3 + 4\Omega^4 \tilde{J}_3 = 0, \tag{26}$$

which is easily solved by the ansatz $\tilde{J}_3(t) = \tilde{J}_3(0) \exp(i\omega t)$ to give

$$\omega^4 - 5\Omega^2 \omega^2 + 4\Omega^4 = 0, \tag{27}$$

for the new effective frequency operators ω .

Hence we obtain $\omega = \pm\Omega$ or $\pm 2\Omega$ and the complete solution for $J_3(t)$ may be expressed in terms of harmonic functions of ωt multiplied by combinations of the initial operators $J_3(0)$, $\dot{J}_3(0)$, $\ddot{J}_3(0)$ and $\dddot{J}_3(0)$, which are easily constructed from previously derived equations. The energy level diagram obtained by Senitzky (1971) for the three-level atom implies that precisely the above frequencies would arise in a calculation of $\langle J_3(t) \rangle$ when the system is at resonance.

4. Higher-order equations

4.1. Three-level atom off resonance

When the three-state system is detuned from resonance ($\alpha = \varepsilon - 1 \neq 0$) the reduction of equation (22) to soluble form is considerably more difficult. Again using $J^2 = 2$ and $J_3^3 = J_3$ we find

$$\ddot{J}_3 + 5\Omega^2 \dot{J}_3 + 4\Omega^4 J_3 = (\frac{3}{2}\alpha\lambda^2)\{A, J_3\} + \Delta, \tag{28}$$

in which $\Omega^2 = \alpha^2 + \lambda^2 (C + \frac{1}{2})$ and we have defined

$$\Delta = 2\Omega^2 [2\alpha A + \lambda^2] - 3\lambda^2 [A^2 + 2\alpha^2]. \tag{29}$$

We next eliminate the anticommutator $\{A, J_3\}$. Differentiating twice more leads to an equation of sixth order involving the quantity $\{A, \ddot{J}_3\}$ into which we substitute equation (14). This gives rise to the anticommutators $\{A, J_3\}$ and $\{A, J_3^2\}$, the first of which may now be eliminated by use of equation (28) itself. The second may be simplified using the definition $A = D + \alpha J_3$ of equation (9d) together with the special properties of the operators J_\pm and J_3 for spin one. After some tedious algebra we arrive at the simple result

$$\{A, J_3^2\} = \alpha J_3 + A. \tag{30}$$

Hence we deduce the sixth-order linear equation

$$d^6 J_3/dt^6 + 6\Omega^2 d^4 J_3/dt^4 + 9\Omega^4 d^2 J_3/dt^2 + [4\Omega^6 - \frac{27}{4}\alpha^2 \lambda^4] J_3 = R, \tag{31}$$

where the constant operator R is defined by

$$R = 9\alpha\lambda^2 A[\alpha A - (\frac{1}{4}\lambda^2)] + \Omega^2 \Delta. \tag{32}$$

It is now straightforward to construct the general solution of equation (31) by considering

$$\tilde{J}_3 = J_3 - 4R/[16\Omega^6 - 27\alpha^2 \lambda^4], \tag{33}$$

which satisfies the homogeneous equation

$$d^6 \tilde{J}_3/dt^6 + 6\Omega^2 d^4 \tilde{J}_3/dt^4 + 9\Omega^4 d^2 \tilde{J}_3/dt^2 + [4\Omega^6 - \frac{27}{4}\alpha^2 \lambda^4] \tilde{J}_3 = 0. \tag{34}$$

The trial solution $\tilde{J}_3(t) = \tilde{J}_3(0) \exp(i\omega t)$ gives the following equation for the effective frequency operators ω :

$$\omega^6 - 6\Omega^2 \omega^4 + 9\Omega^4 \omega^2 - [4\Omega^6 - (\frac{27}{4}\alpha^2 \lambda^4)] = 0, \tag{35}$$

which may be rewritten in the form

$$[3\omega/2\Omega - \omega^3/2\Omega^3]^2 = [1 - 27\alpha^2 \lambda^4/16\Omega^6]. \tag{36}$$

The solutions of this equation are found to be

$$\omega_m = \pm 2\Omega \sin[\frac{1}{3}\{m\pi + \cos^{-1}(3\sqrt{3} \alpha\lambda^2/4\Omega^3)\}] \tag{37}$$

where $m = \pm 1, 0$. The frequencies thus obtained are consistent with the eigenvalue spectrum of a three-level atom coupled to the radiation field.

4.2. Resonant four-level atom

As a final example we consider radiation in resonance with a four-state system ($J = \frac{3}{2}$). To treat this we must return to equation (22), put $\alpha = 0$, and use the same technique as before. That is, we continue to differentiate and at each stage eliminate unwanted operators by use of the lower-order equations. Eventually the process stops when we reach an equation with a quartic nonlinearity which may be reduced by means of the operator identities $J^2 = \frac{15}{4}$ and

$$(J_3^2 - \frac{1}{4})(J_3^2 - \frac{9}{4}) = 0. \tag{38}$$

The algebraic manipulations are long and involved and we give here only the final result, which is the eighth-order linear equation

$$d^8 J_3/dt^8 + 15\Omega^2 d^6 J_3/dt^6 + 9\Omega^4 [7 - 3\lambda^4/4\Omega^4] d^4 J_3/dt^4 + 5\Omega^6 [17 - 9\lambda^4/2\Omega^4] d^2 J_3/dt^2 + 9\Omega^8 [4 - 7\lambda^4/4\Omega^4 - 9\lambda^8/4\Omega^8] J_3 = \Lambda, \tag{39}$$

in which $\Omega^2 = \lambda^2(C + \frac{1}{2})$ and we define

$$\Lambda = (\frac{21}{16}\lambda^6)[135\Omega^2 + 2A^2] + (\frac{27}{4}\lambda^2\Omega^4)[5\Omega^2 - 4A^2]. \tag{40}$$

In terms of the variable

$$\nu = [1 + (9\lambda^4/16\Omega^4)]^{1/2} - 1 \tag{41}$$

the effective frequency operators are given by

$$\begin{aligned}
 \omega_1 &= \pm \Omega[1 - 4\nu]^{1/2} \\
 \omega_2 &= \pm \Omega[9 + 4\nu]^{1/2} \\
 \omega_3 &= \pm (\tfrac{1}{2}\Omega)[(9 + 4\nu)^{1/2} - (1 - 4\nu)^{1/2}] \\
 \omega_4 &= \pm (\tfrac{1}{2}\Omega)[(9 + 4\nu)^{1/2} + (1 - 4\nu)^{1/2}]
 \end{aligned}
 \tag{42}$$

and agree with the energy eigenvalue spectrum

$$E = \pm (\tfrac{1}{2}\Omega)[1 - 4\nu]^{1/2}, \pm (\tfrac{1}{2}\Omega)[9 + 4\nu]^{1/2}
 \tag{43}$$

found by Senitzky (1971). The general solution of equation (39) is of course a linear combination of the solutions with frequencies ω_n , $n = 1, 2, 3, 4$.

5. Summary

In this paper we have demonstrated by example that it is possible to solve the second-order nonlinear Heisenberg equation for the model of a few-level atom in resonant or non-resonant interaction with a single mode radiation field. The method can clearly be applied systematically for an atomic system with an arbitrary number of equally spaced states. It is conceivable that a general recursive formulation remains to be discovered.

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